UC SANTA BARBARA

Flow Network Problems on the n-torus with Asymmetric Coupling

Robin Delabays

robindelabays@ucsb.edu

Joint work with
S. Jafarpour and F. Bullo

Acknowledgment

Saber Jafarpour (Georgia Tech)

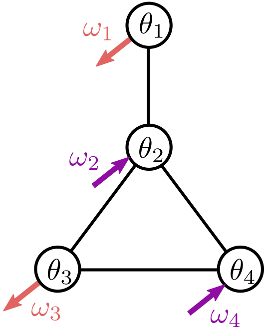
Francesco Bullo (UCSB)

Financial support:

Swiss National Science Foundation

"Flow Network Problems..." or Synchronization

Graph: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, Adjacency matrix: $A = (a_{ij})_{i,j=1}^n$.



Flow network:

$$\omega_i = \sum_{j=1}^n a_{ij} h_{ij}(\theta_i - \theta_j)\,.$$

Synchronization: $\dot\theta_i = \dot\theta_j$,

$$\dot\theta_i = \omega_i - \sum_{j=1}^n a_{ij} h_{ij}(\theta_i - \theta_j)\,.$$

Number of solutions?

"...Asymmetric Coupling"

The **Kuramoto** model:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)\,.$$

Symmetric: $\sin(\theta_i - \theta_j) = -\sin(\theta_j - \theta_i)$.

The **Kuramoto-Sakaguchi** model:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \left[\sin(\theta_i - \theta_j - \phi) + \sin\phi\right]\,.$$

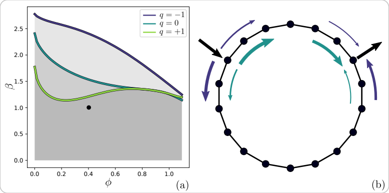
Asymmetric: $\sin(\theta_i - \theta_j - \phi) + \sin\phi \neq -[\sin(\theta_j - \theta_i - \phi) + \sin\phi]$.

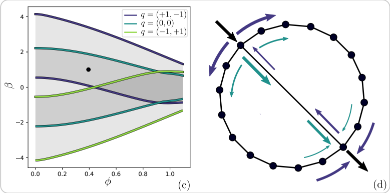
H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, *Local and Global Self-Entrainments in Oscillator Lattices*, Prog. Theor. Phys., **77**(5):1005–1010 (1987).

"...Asymmetric Coupling" (bis)

Asymmetric coupling:

$$\omega_i = \sum_{j=1}^{n} a_{ij} h_{ij}(\theta_i - \theta_j)\,, \qquad\qquad h_{ij}(x) \neq -h_{ji}(-x)\,.$$



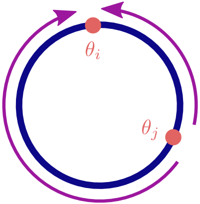


"...the *n*-torus..." and winding cells

Oscillators: $\theta_i \in \mathbb{S}^1 \simeq [-\pi, \pi)$.

System state: $\boldsymbol{\theta} \in (\mathbb{S}^1)^n = \mathbb{T}^n$.

Coupling: $h_{ij}(x) = h_{ij}(x + 2\pi)$.

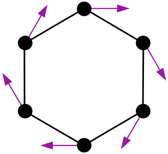


Counterclockwise difference:

$$d_{\mathrm{cc}}(\theta_i, \theta_j) = \mathrm{mod}(\theta_i - \theta_j + \pi, 2\pi) - \pi \in [-\pi, \pi)$$
$$= \theta_i - \theta_j + 2\pi k\,.$$

S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo, *Flow and Elastic Networks on the n-torus*, to appear in SIAM Review, arXiv:1901.11189 (2021).

"...the n-torus..." and winding cells (bis)



Around a cycle $\sigma = (1, 2, ..., \ell, 1)$:

Winding number:

$$\sum_{i=1}^{\ell} (\theta_i - \theta_{i+1}) = 0 \qquad \rightarrow \qquad \sum_{i=1}^{\ell} d_{\mathrm{cc}}(\theta_i, \theta_{i+1}) = 2\pi q_\sigma \,.$$

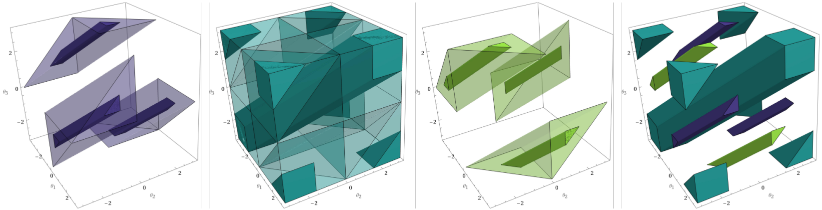
Multiple cycles $\Sigma = \{\sigma_1, ..., \sigma_c\}$, **winding vector**:

$$q_\Sigma(\boldsymbol{\theta}) = (q_{\sigma_1}, ..., q_{\sigma_c})^\top \in \mathbb{Z}^c \,.$$

"...the *n*-torus..." and winding cells (ter)

Winding cells:

$$\Omega_u^\Sigma = \{\boldsymbol{\theta} \in \mathbb{T}^n \colon q_\Sigma(\boldsymbol{\theta}) = u\} \ .$$



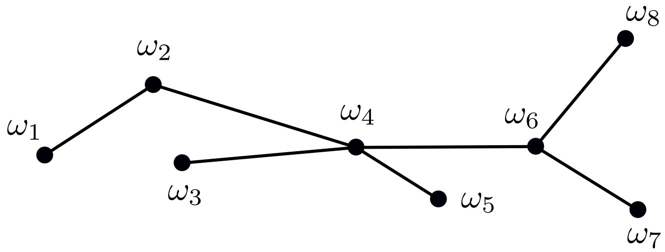
S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo, *Flow and Elastic Networks on the n-torus*, to appear in SIAM Review, arXiv:1901.11189 (2021).

Acyclic Graphs

Theorem

On acyclic graphs, the Flow Network Problem has at most one solution.

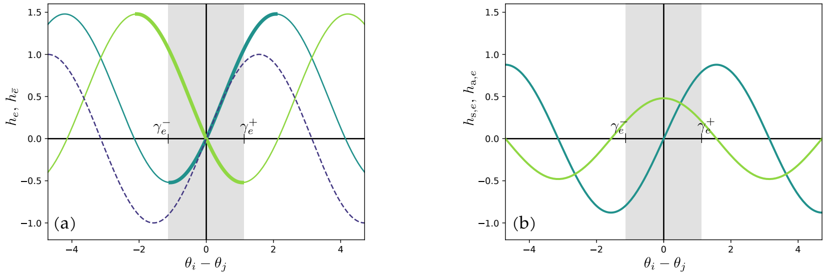
Proof:



General Graphs

Symmetric and asymmeric parts of the coupling:

$$h_{\mathrm{s},e}(x) = \frac{h_e(x) - h_{\bar{e}}(-x)}{2} = \cos\phi \sin x\,, \qquad h_{\mathrm{a},e}(x) = \frac{h_e(x) + h_{\bar{e}}(-x)}{2} = \sin\phi \cos x\,.$$



General Graphs (bis)

Theorem

There is at most one solution of the Flow Network Problem in each winding cell, provided the couplings are **not too asymmetric**.

Proof:

- Define an fixed point map;
- Show that it is contracting.

"...not too asymmetric...": $\lambda_2[L_\mathrm{s}(\boldsymbol{\theta})] > \max_i[D_\mathrm{a}(\boldsymbol{\theta})]_{ii}$.

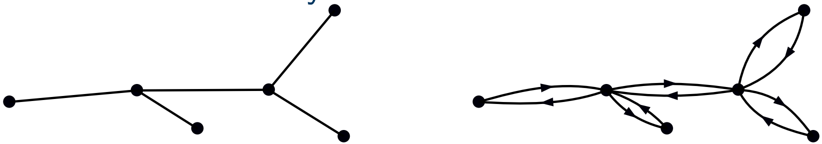
S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo, *Flow and Elastic Networks on the n-torus*, to appear in SIAM Review, arXiv:1901.11189 (2021).

Wrap-up

1. Framework for any network flow problem;
2. At most uniqueness in each winding cell.

Thank you!

Vectorial form of the dynamics



$$B_\mathrm{b} = [B, -B], \qquad B_\mathrm{o} = [B_\mathrm{b}]_+ .$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - B_\mathrm{o}\left[\sin\left(B_\mathrm{b}\boldsymbol{\theta} - \phi\right) + \sin\phi\right]$$

Iteration map:

$$\begin{aligned} S_\delta\colon \mathbb{R}^m &\to \mathbb{R}^m \\ \boldsymbol{\Delta} &\mapsto \boldsymbol{\Delta} - \delta B^\top L^\dagger\left[B_\mathrm{o} h(\boldsymbol{\Delta}) - \boldsymbol{\omega}\right] . \end{aligned}$$