Conformal field theory: physics without scales

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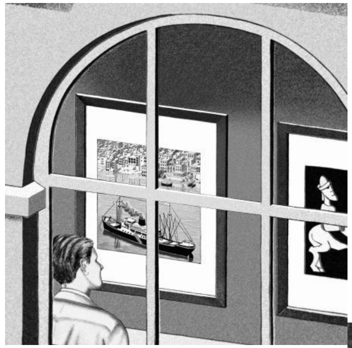
MathPhys Seminar — November 2, 2023

...as in **CONFORMAL TRANSFORMATION**



M.C. Escher
(1898–1972)

CONFORMAL FIELD THEORY

...as in **QUANTUM FIELD THEORY**

- particle physics
- statistical and condensed matter physics
- string theory (quantum theory of gravity)

A hot topic

The Large N limit of superconformal field theories and supergravity

Juan Martin Maldacena (Harvard U.)
Nov, 1997

21 pages
Published in: *Int.J.Theor.Phys.* 38 (1999) 1113-1133 (reprint), *Adv.Theor.Math.Phys.* 2 (1998) 231-252
e-Print: hep-th/9711200 [hep-th]
DOI: 10.4310/ATMP.1998.v2.n2.a1 (publication), 10.1023/A:1026654312961 (reprint)
Report number: HUTP-97-A097, HUTP-98-A097
View in: OSTI Information Bridge Server, ADS Abstract Service, AMS MathSciNet

pdf | cite | claim | reference search | 19,015 citations

- In physics:
  - Breakthrough Prize in Fundamental Physics
    - 2013: Alexander Polyakov
    - 2024: John Cardy & Alexander Zamolodchikov
  - AdS/CFT correspondence: most cited paper in high-energy physics

- In mathematics:
  - Fields medals:
    - 2010: Stanislav Smirnov
    - 2022: Hugo Duminil-Copin & Maryna Viazovska

Hugo Duminil-Copin

For solving longstanding problems in the probabilistic theory of phase transitions in statistical physics, especially in dimensions three and four.

citation | video | popular scientific exposition | CV/publications
interview | laudatio | proceedings | Plus magazine! article (intro)

Maryna Viazovska

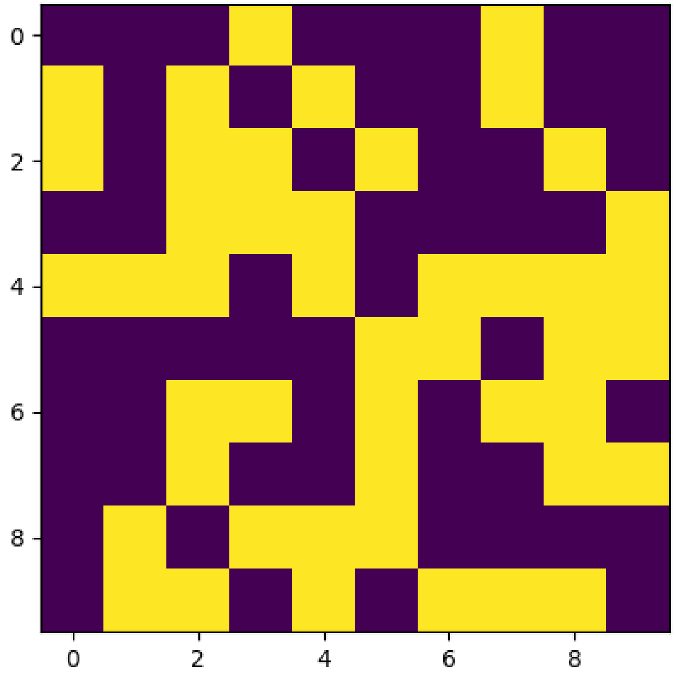
For the proof that the E_8 lattice provides the densest packing of identical spheres in 8 dimensions, and further contributions to related extremal problems and interpolation problems in Fourier analysis.

citation | video | popular scientific exposition | CV/publications
interview | laudatio | proceedings | Plus magazine! article (intro)

Part 1

The Ising model

The Ising model



2-dimensional, n x n lattice

"Spins": $\sigma_i = \pm 1$

Energy density: $\varepsilon_i = -\frac{1}{2} \sum_{|i-j|=1} \sigma_i \sigma_j$

Energy: $H = - \sum_{|i-j|=1} \sigma_i \sigma_j$

Ising dynamics

Rules of the game:

- Pick a lattice site at random
- Compute the energy cost ΔE of flipping the spin $\sigma_i \to -\sigma_i$
  - If $\Delta E < 0$ flip the spin
  - If $\Delta E \geq 0$ flip it anyway with probability $P = e^{-\beta \Delta E}$
- Repeat

$\beta = \frac{1}{k_B T}$ inverse of temperature in thermodynamics

Correlation length in the Ising model

- Correlation generally decreases exponentially with the distance

$$\langle \sigma_i \sigma_j \rangle \propto \exp\left(-\frac{|i-j|}{\xi(\beta)}\right)$$

- But correlation length diverges at critical value $\beta_c = \frac{\log(1+\sqrt{2})}{2} \cong 0.441$

Lars Onsager (1944)

- Correlation at criticality is given by a power of the distance:

$$\langle \sigma_i \sigma_j \rangle \propto \frac{1}{|i-j|^{1/4}}$$

Scaling at criticality

(Smirnov's Fields medal)

Continuum limit: $n \to \infty \qquad \sigma_i \to \sigma(x)$

$$\langle \sigma(x)\sigma(y) \rangle \propto \frac{1}{|x-y|^{2\Delta_\sigma}}$$

$$\langle \varepsilon(x)\varepsilon(y) \rangle \propto \frac{1}{|x-y|^{2\Delta_\varepsilon}}$$

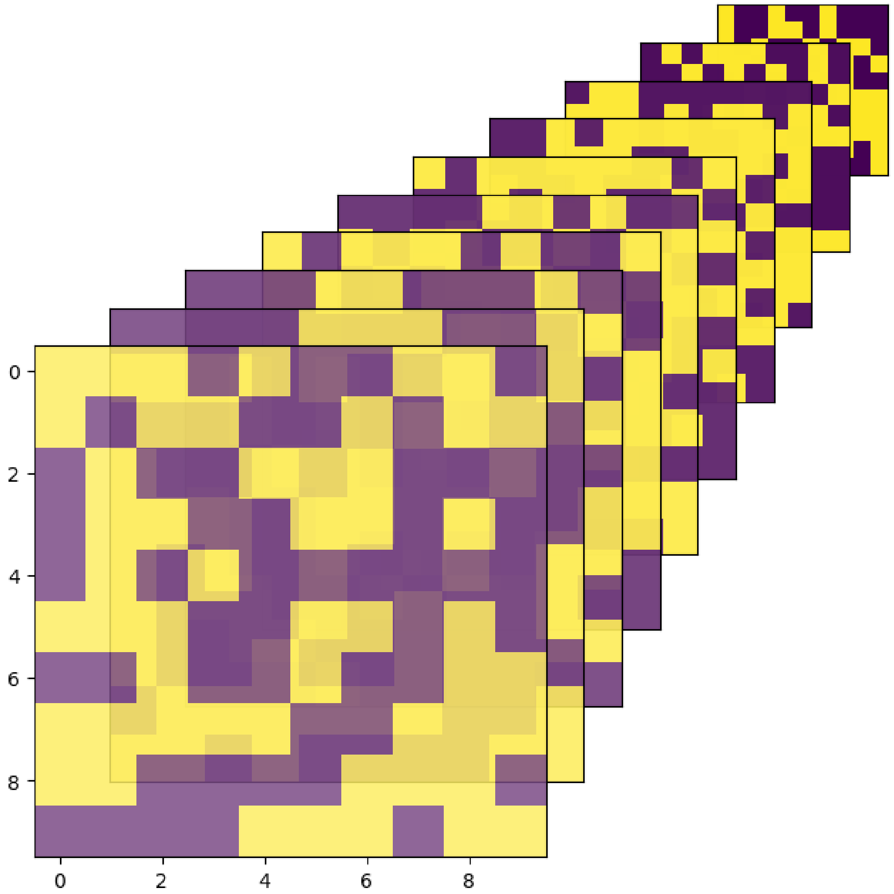
$$\Delta\sigma = \frac{1}{8} \qquad \Delta\varepsilon = 1$$

$$\langle \sigma(x)\varepsilon(y)\varepsilon(z) \rangle \propto \frac{1}{|x-y|^{\Delta_\sigma}\,|x-z|^{\Delta_\sigma}\,|y-z|^{2\Delta_\epsilon - \Delta_\sigma}}$$

$$\langle \varepsilon(x)\varepsilon(y)\varepsilon(z) \rangle \propto \frac{1}{|x-y|^{\Delta_\varepsilon}\,|x-z|^{\Delta_\varepsilon}\,|y-z|^{\Delta_\varepsilon}}$$

CONFORMAL FIELD THEORY

The Ising model in 3d



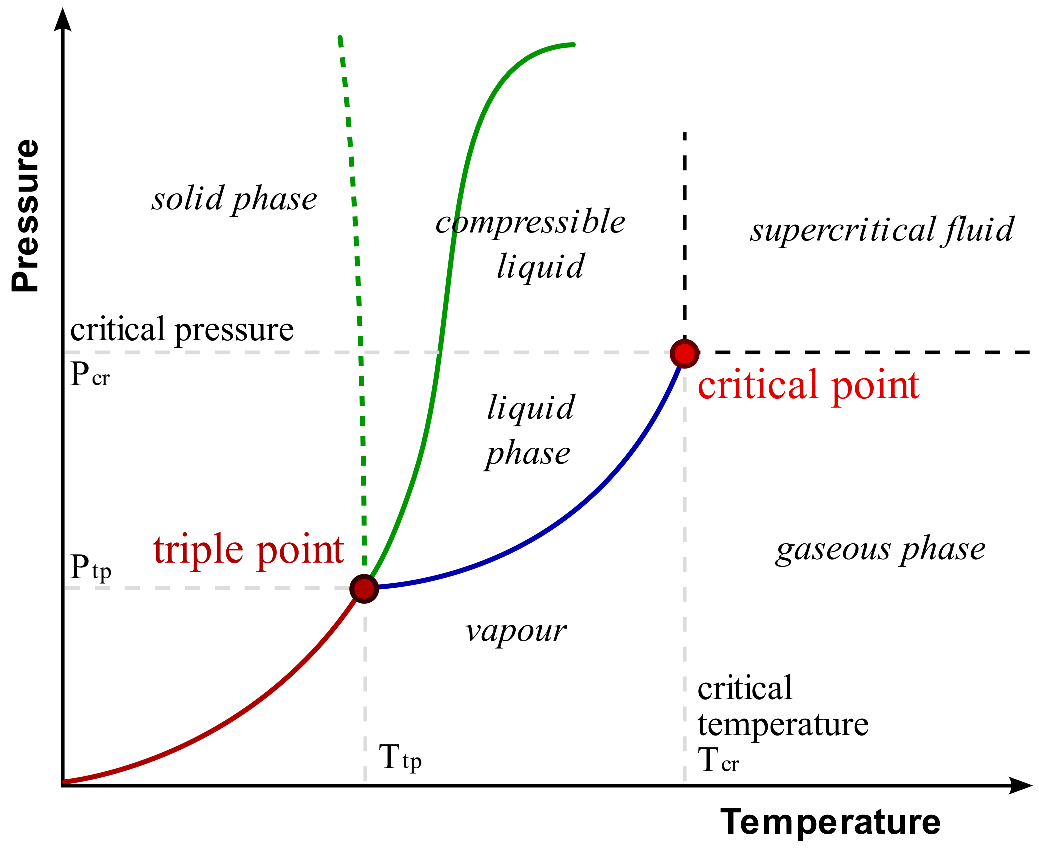
- Unlike 2d, exact solution is unknown
- Still, critical point observed (Duminil-Copin's Fields medal)
- Scaling laws consistent with CFT, with

$$\Delta_\sigma \cong 0.518$$

$$\Delta_\varepsilon \cong 1.413$$

Universality of critical phenomena

- Ising model describes ferromagnets
- But not only: critical exponents matching with critical point of liquid-gas transition, e.g. in water
- Connection?
  - ~~scale~~ conformal invariance
  - only 2 relevant parameters



Part 2

Conformal symmetry

A fundamental principle in physics

The laws of nature do not depend on the reference frame

Symmetry under

- Translations: $x_i \to x_i + a_i$
- Rotations: $x_i \to (R \cdot x)_i \qquad R^T \cdot R = 1$

These are the most general coordinate transformations that preserve the notion of distance:

$$|x - y| = \sqrt{\sum_i (x_i - y_i)^2}$$

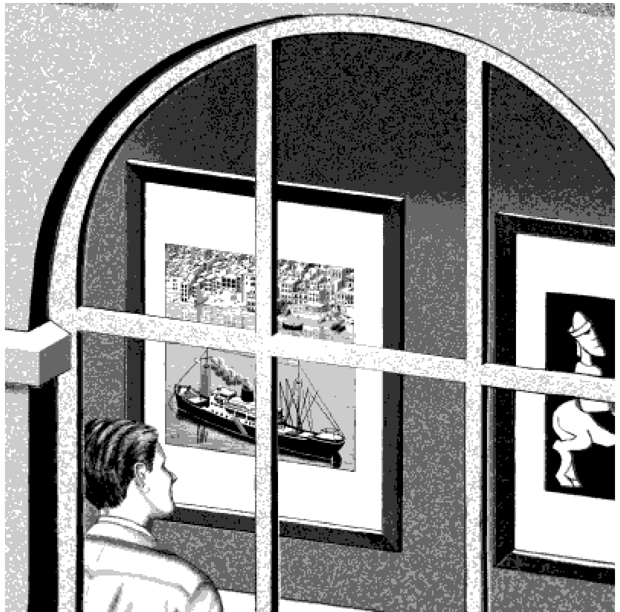
Scale symmetry

- If a system does not have a reference scale, two different observers could describe it in terms of different systems of units

- Symmetry under scale transformations: $x_i \to \lambda x_i$

- Preserves the notion of distance up to **global** rescaling:

$$|x - y| \to \lambda \, |x - y|$$



de Smit and Lenstra, *Notices of the AMS* vol. 50 num 4 (446)

Special conformal symmetry

- Observers could even use different units at different points in space

- **Local** rescaling of the notion of distance $|x-y| \to \Omega(x)\Omega(y)\,|x-y|$

- Solved by the **special conformal transformation** $x_i \to \dfrac{x_i - x^2 b_i}{1 - 2b\cdot x + b^2 x^2}$

$$\Omega(x) = \left(1 - 2b\cdot x + b^2 x^2\right)^{-1/2}$$

- Complete, up to:
  - Inversion $x_i \to x_i/x^2$
  - Many more specifically in 2d

The conformal group

Translations	3
Rotations	3
Scale transformations	1
Special conformal transformations	3
Total	**10**

- Conformal transformations in 3d form a group, isomorphic to SO(4,1)
- They map spheres onto spheres, and circles onto circles (including the line, a circle of infinite radius)
- 3 points can be mapped onto any 3 other points
- Infinity is treated like a point (conformal group acts on $\mathbb{R}^3 \cup \{\infty\}$)

Conformal field theory

- Require fields to transform covariantly, e.g. for a scalar field

$$\sigma(x) \to \Omega(x)^{-\Delta_\sigma} \sigma(x)$$

- 2-point functions:

$$\langle \sigma(x)\sigma(y) \rangle = \frac{1}{|x-y|^{\Delta_\sigma}} \qquad \langle \sigma(x)\varepsilon(y) \rangle = 0$$

- 3-point functions:

$$\langle \sigma(x)\varepsilon(y)\phi(z) \rangle = \frac{C_{\sigma\varepsilon\phi}}{|x-y|^{\Delta_\sigma+\Delta_\varepsilon-\Delta_\phi}\, |x-z|^{\Delta_\sigma+\Delta_\phi-\Delta_\varepsilon}\, |y-z|^{\Delta_\varepsilon+\Delta_\phi-\Delta_\sigma}}$$

Quantumness

- The fields are actually operators acting on states
- Unique vacuum state, conformally invariant: $|0\rangle$
- All other states are obtained acting with operators:

$$\sigma(x)\,|0\rangle\,, \quad \varepsilon(x)\,|0\rangle\,, \quad \ldots$$

- Correlation functions compute the norm of states:

$$\langle 0|\,\sigma(x)\sigma(y)\,|0\rangle$$

4-point correlations in a CFT

$$u = \frac{|x-y|^2 |z-w|^2}{|x-z|^2 |y-w|^2}$$

$$v = \frac{|x-w|^2 |y-z|^2}{|x-z|^2 |y-w|^2}$$

- Fixed by conformal symmetry up to two "cross-ratios":

$$\langle 0| \sigma(x)\sigma(y)\sigma(z)\sigma(w) |0\rangle = \frac{G(u,v)}{|x-y|^{2\Delta_\sigma} |z-w|^{2\Delta_\sigma}}$$

- Can be expanded summing over all possible intermediate states

$$\langle 0| \sigma\sigma\sigma\sigma |0\rangle \sim \sum_\phi \langle 0| \sigma\sigma\phi |0\rangle \langle 0| \phi\sigma\sigma |0\rangle$$

$$\Rightarrow \qquad G(u,v) = 1 + \sum_\phi C^2_{\sigma\sigma\phi}\, g_\phi(u,v)$$

fixed by conformal symmetry in terms of Δ_ϕ

Conformal bootstrap

- Different ways of summing over states give different sums

$$1 + \sum_\phi C^2_{\sigma\sigma\phi} g_\phi(u,v) = \left(\frac{u}{v}\right)^{\Delta_\sigma} \left[1 + \sum_\phi C^2_{\sigma\sigma\phi} g_\phi(v,u)\right]$$

- Maybe not all spectra of operators are consistent with this?

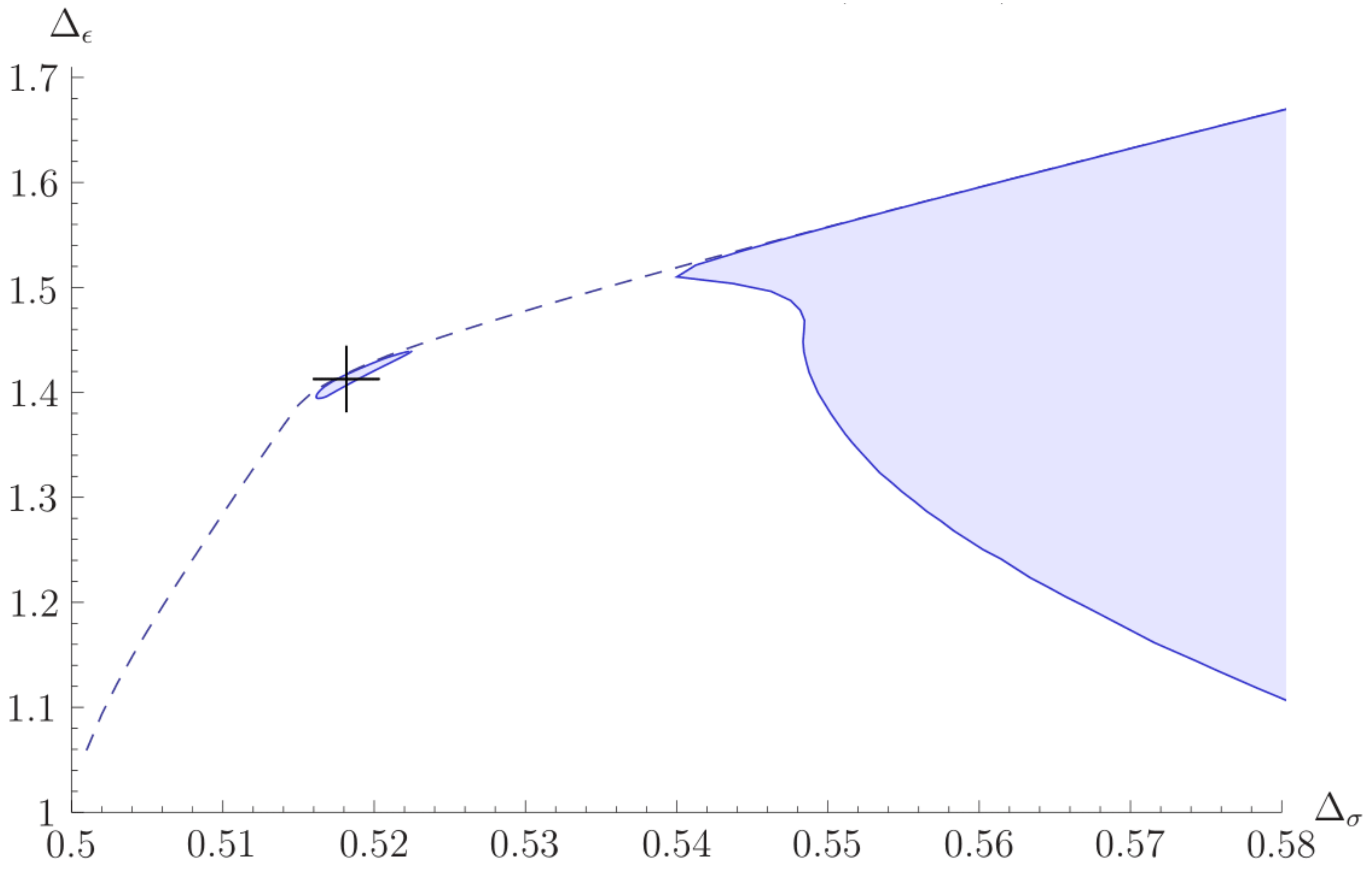
Polyakov (1974)

Idea: find $\Delta_\sigma, \Delta_\phi$ such that the following equation is inconsistent:

$$u^{\Delta_\sigma} - v^{\Delta_\sigma} = \sum_\phi C^2_{\sigma\sigma\phi} \left[v^{\Delta_\sigma} g_\phi(u,v) - u^{\Delta_\sigma} g_\phi(v,u)\right]$$

Rattazzi, Rychkov, Tonni, Vichi (2008)

Recovering the 3d Ising model



Assumptions:

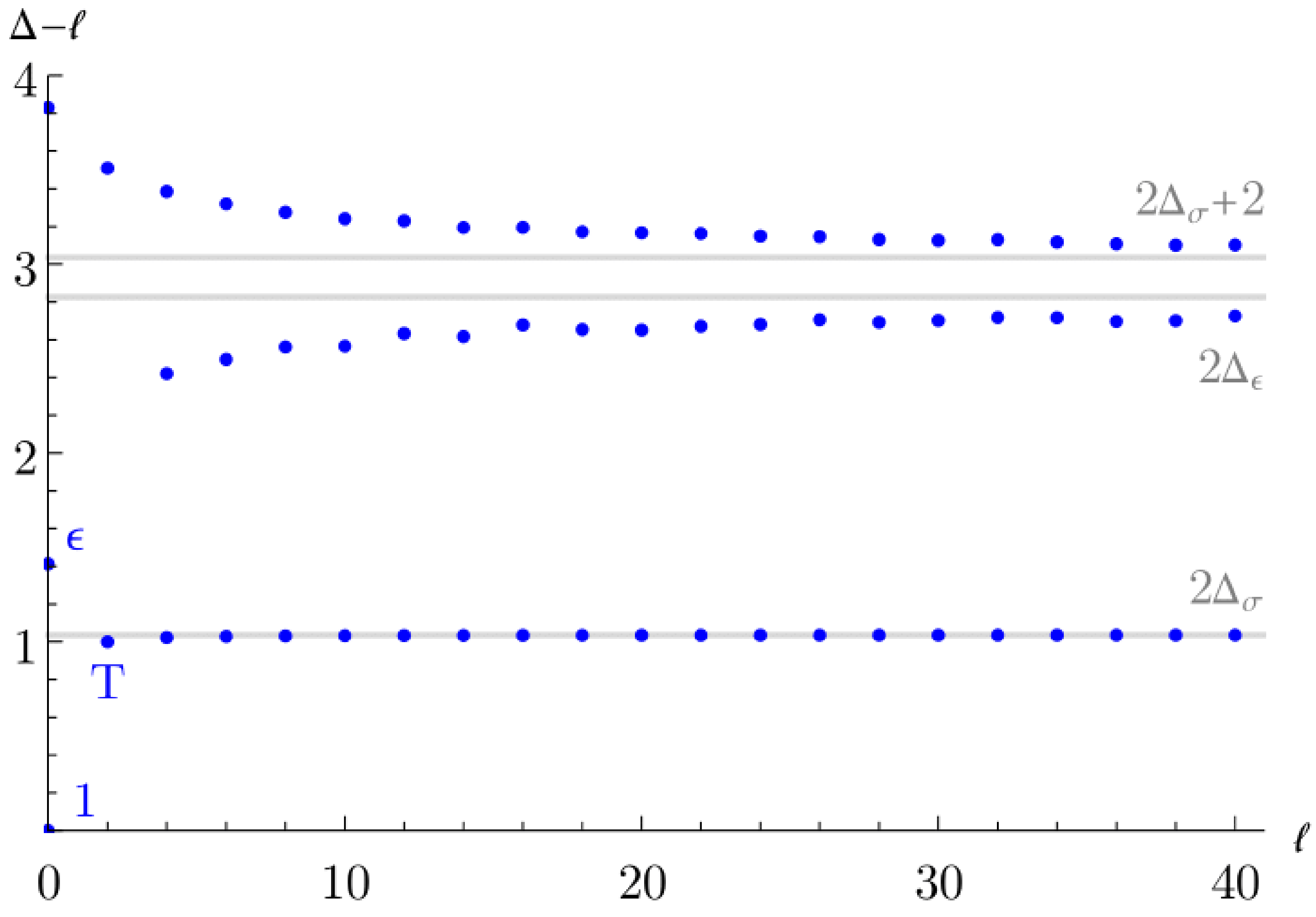
- Conformal symmetry
- 2 relevant operators
- $\mathbb{Z}_2$ symmetry

Most precise determination of scaling dimension with rigorous error bars (2016):

$$\Delta_\sigma = 0.5181489(10)$$

F. Kos, D. Poland, and D. Simmons-Duffin, JHEP 11 (2014) 109

Operator spectrum of the 3d Ising model



My figure, data from D. Simmons-Duffin, JHEP 03 (2017) 086

Conformal bootstrap summary

- The space of all CFTs is sparse (explaining universality)

  $\Rightarrow$ Is it possible to classify all of them?

- Quantum field theory formulated in a rigorous way

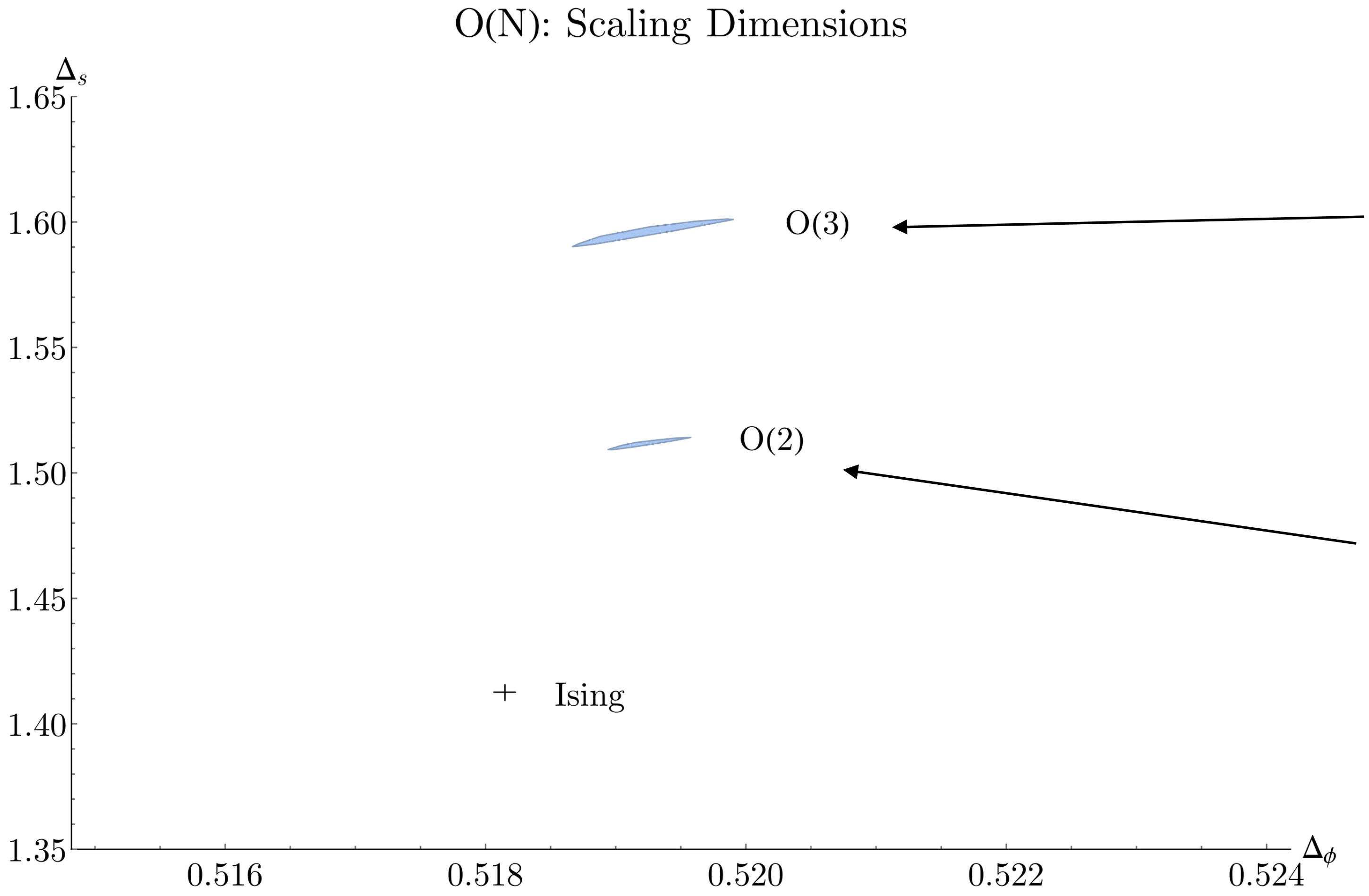
  $\Rightarrow$ Vertex Operator Algebras

- Fun fact: analytic functionals giving the optimal bound are identical to those of the sphere packing problem (Viazovska's Fields medal)

Part 3

A few applications

O(N) models



Heisenberg magnets

Superfluid transition of liquid helium

Kos, Poland, Simmons-Duffin, Vichi, JHEP 11 (2015) 106

4 space-time dimensions

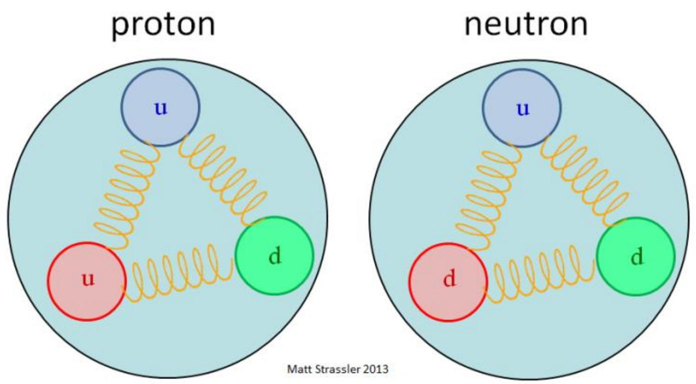
- Same logic applies to space-time coordinates (3 space + 1 time dimension)

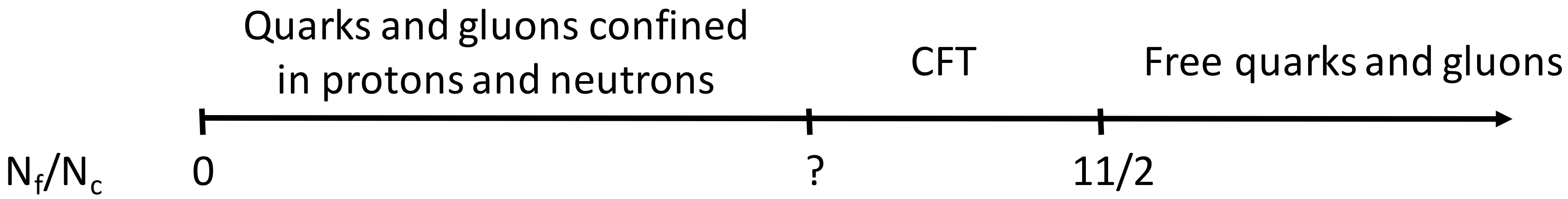
- Bigger conformal group:
  - In 3d $\mathbb{R}^3 \rtimes SO(3) \subset SO(4,1)$
  - In 4d $\mathbb{R}^{3,1} \rtimes SO(3,1) \subset SO(4,2)$

- Remark: quantum field theory in 4d is the language of particle physics, but no particles in CFT (except in free theories)

The conformal window of QCD

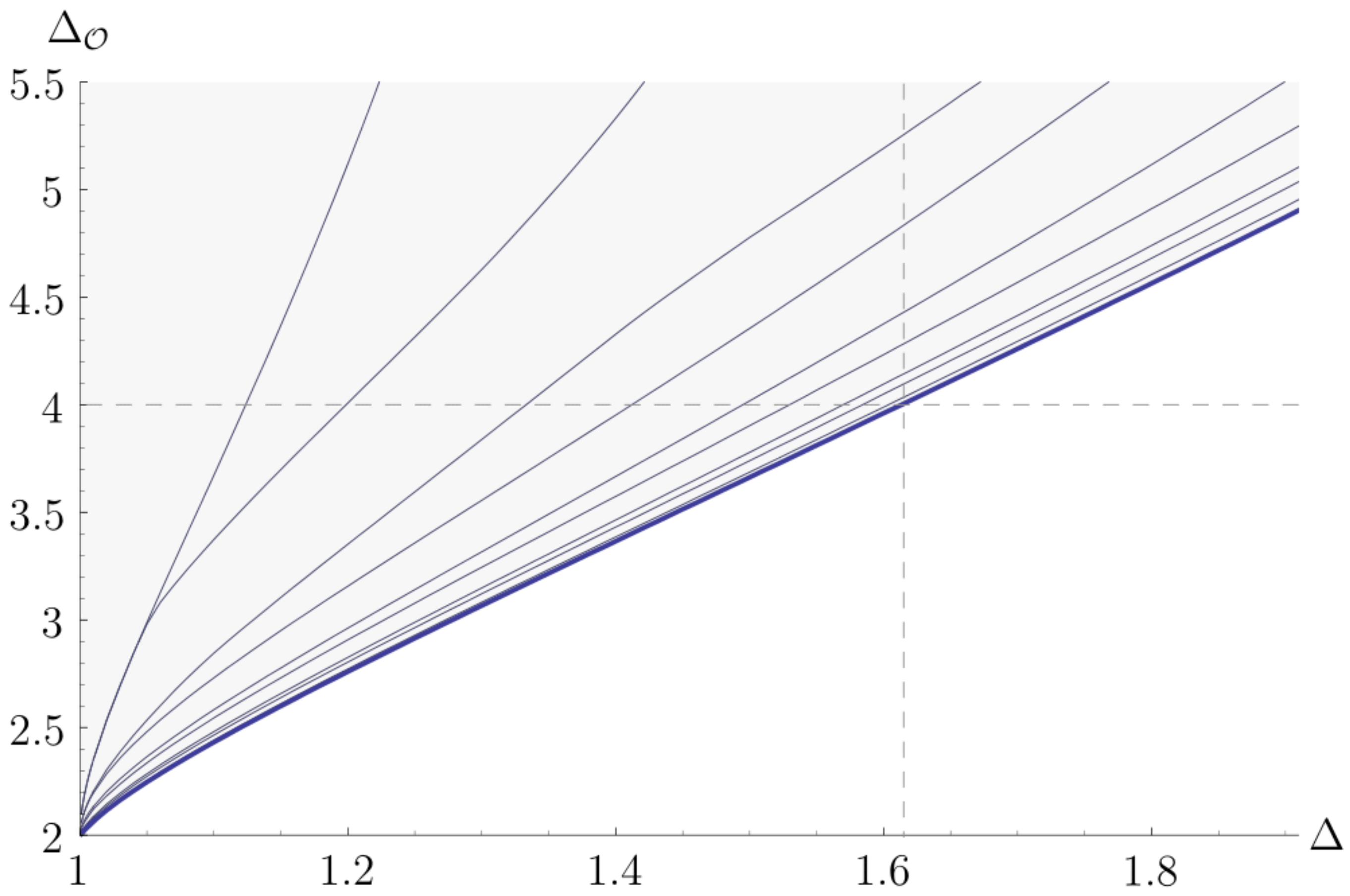
- "Quantum chromodynamics": the theory of quarks and gluons
- In nature, 3 colors (N_c) and 6 flavors of quarks (N_f)
- Thought experiment: what about other N_c and N_f?





No conclusive results in 4d so far

- Bounds on scalar operators mostly featureless
- Difficult to get bounds on operators with spin and "flavour" symmetries
- Gauge symmetry?
- Exception: supersymmetric theories



D. Poland, D. Simmons-Duffin and A. Vichi, JHEP 05 (2012) 110

Towards a classification?

- Zoo of CFTs in 2 and 3 dimensions
- Indirect arguments in favor of CFTs in 4 dimension
- Some supersymmetric theories in 4, 5 and 6 dimensions
- Nothing interesting above 6 dimensions?