Power Grids as Complex Oscillator Networks

Frank Hellmann and Jakob Niehues

work with

Jakob Niehues, Anna Büttner, Hans Würfel, Nina Kastendiek
Johannes Schiffer, Robin Delabays, and more

Sion
April 2026

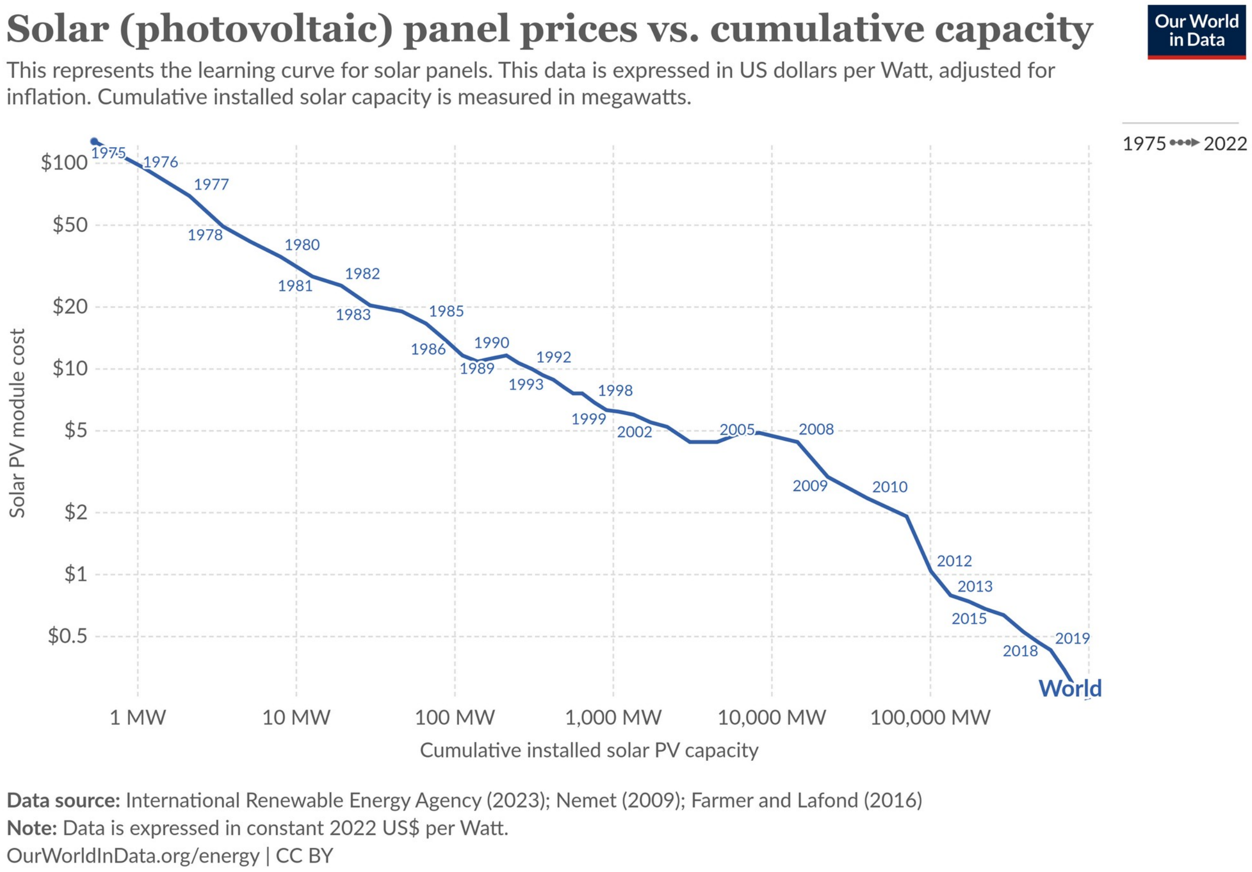
The power grid is a complex and dynamical system

NASA Earth Observatory
Black Marble

Power Grid as Complex Oscillators

The new power economy:

How well do technologies synergize with free solar.

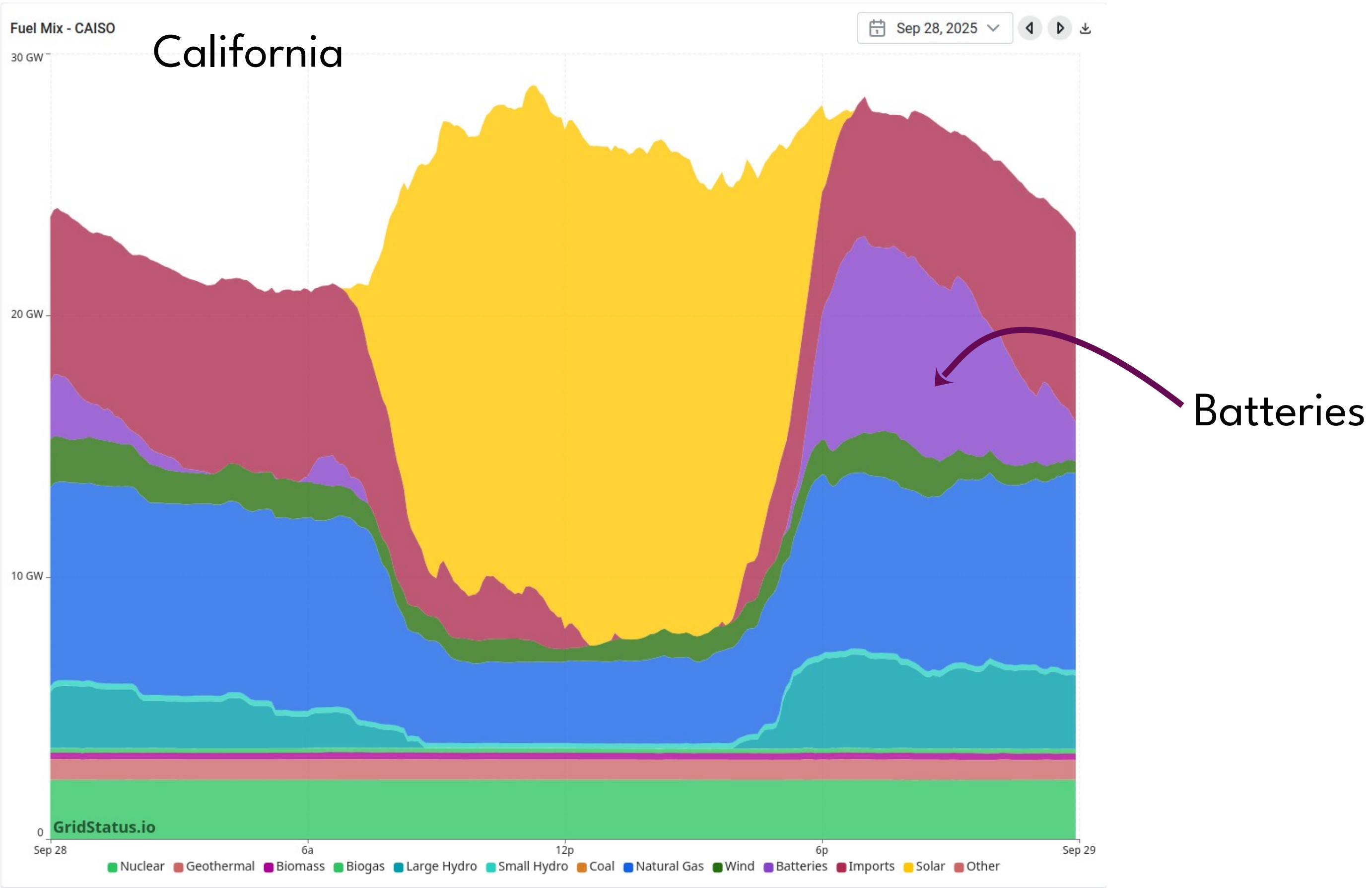


https://ourworldindata.org/grapher/solar-pv-prices

Power Grid as Complex Oscillators

The new power economy:

How well do technologies synergize with free solar.



https://www.gridstatus.io

Power Grid as Complex Oscillators

The new technological reality:

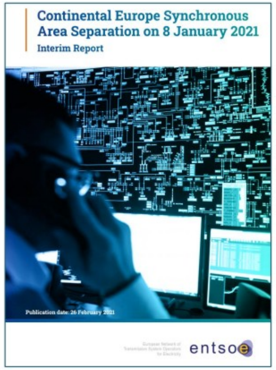
How do we guarantee the stable synchronous state that is required for power flow?

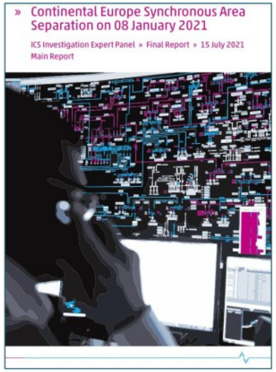
e.g. when cascading failures that split the system

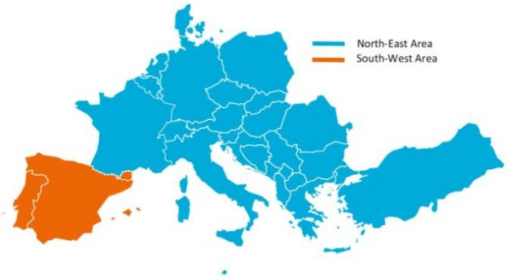
Teilnetzbildungen

Aktuelle Großstörung

- System Split am 8. Januar 2021
- System Split am 24. Juli 2021







Power Grid as Complex Oscillators

… or emergent instabilities take whole countries offline in seconds.

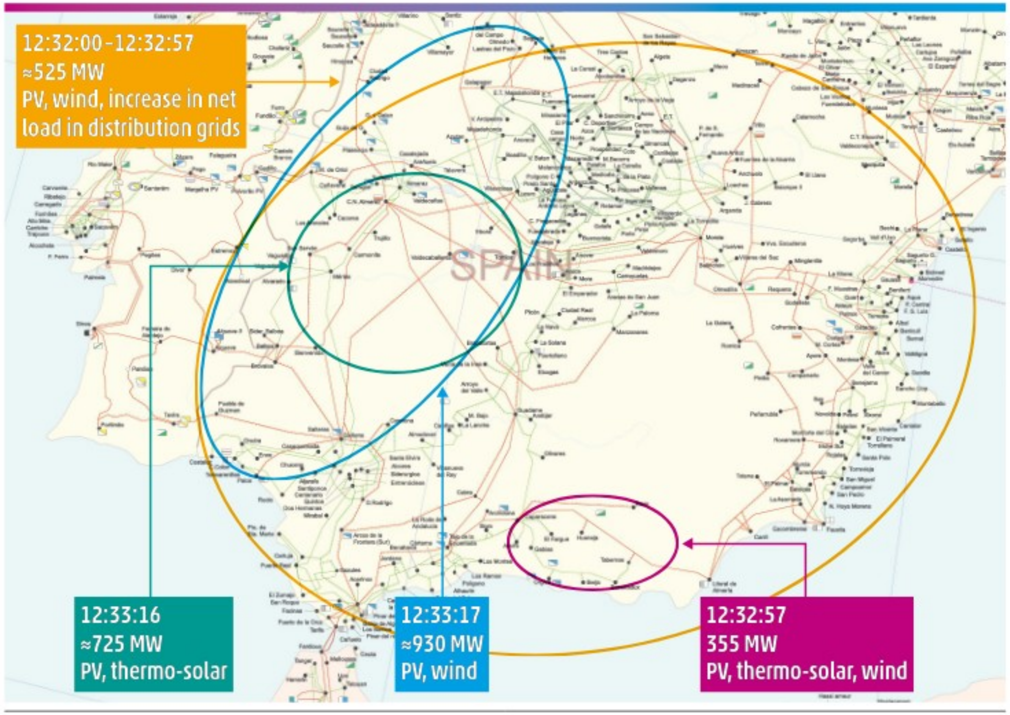
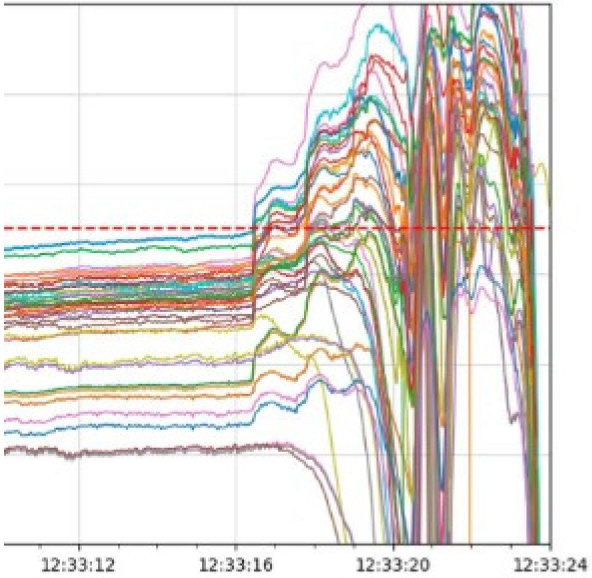
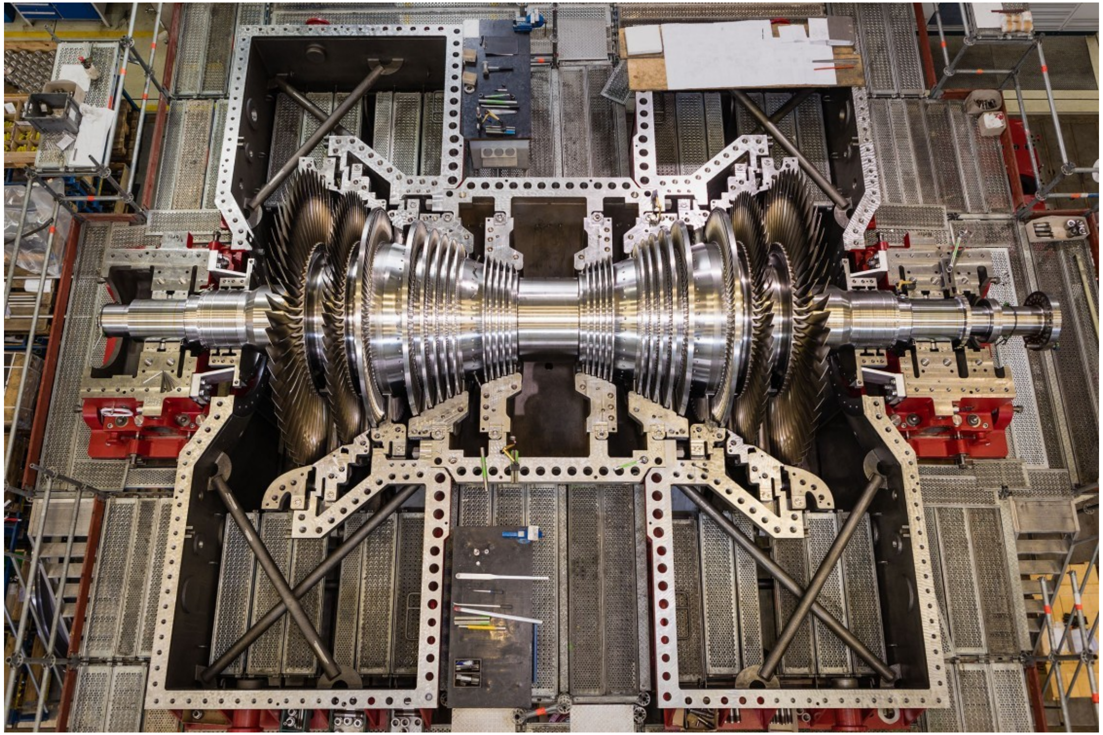


Figure 1-2: Areas of major generation disconnection events in Spain until 12:33:18



Power Grid as Complex Oscillators

Challenge: New actors.



We are replacing thousands of heavy rotating turbines with well understood dynamics by **millions** of programmable inverters with a wide variety of designs and behaviors.

Siemens Energy

Power Grid as Complex Oscillators

We are replacing thousands of heavy rotating turbines with well understood dynamics by **millions** of programmable inverters with a wide variety of designs and behaviors.

Back to first principles and physics.

Part I – A complex systems perspective on dynamics

Back to first principles and physics.

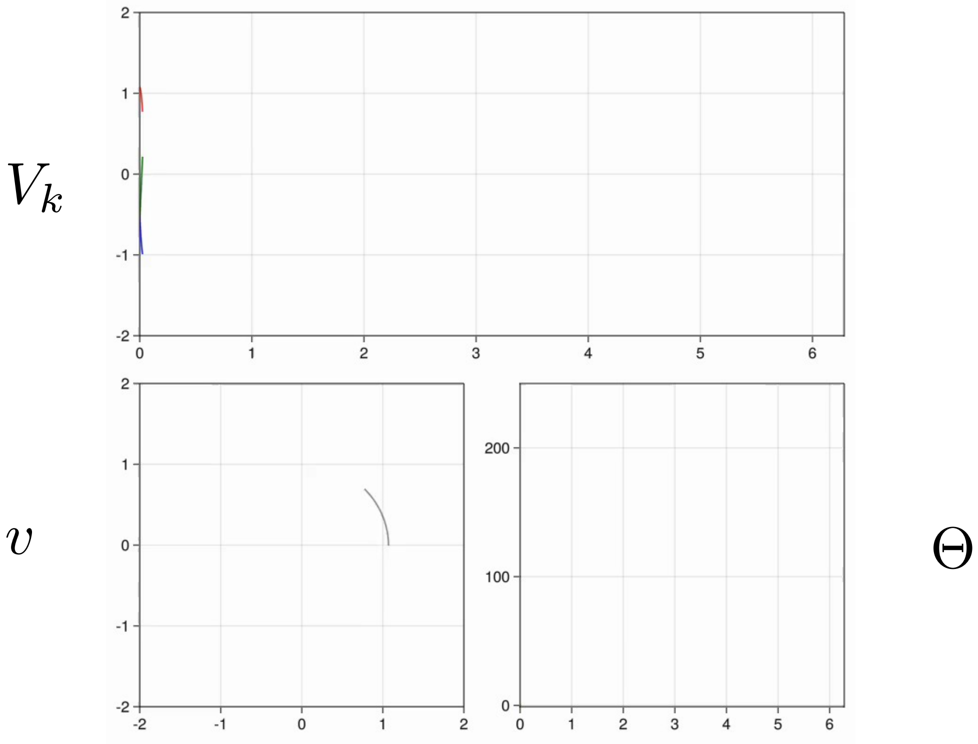
Power Grid as Complex Oscillators

A power grid has three phases, that are typically in balance. We can translate these oscillations into the (complex) plane.

$$V \cdot 1 = \sum_{k=1\ldots3} V_k = 0$$

$$V_k = \mathrm{Re}\left(e^{i\frac{2\pi k}{3}} v(t)\right)$$

$$\Theta = \ln(v)$$



Power Grid as Complex Oscillators
Stuart Landau:

The simplest oscillator that creates a nice stable rotation is Stuart-Landau:

$$\dot{v} = v(j\Omega + c(A - |v|^2)$$

$$\dot{\Theta} = j\Omega + c(A - |v|^2)$$

$$\approx j\Omega - cA(2\Re(\Theta) - \ln(A))$$

Here the symmetry of the desired limit-cycle extends to the whole state space, can we assume this for grid dynamics?

Power Grid as Complex Oscillators
Stuart Landau:

Yes! Typical assumption in engineering models: in a “corotating frame” the dynamics remain independent of time.

This is equivalent to the existence of a U(1) symmetry of the full dynamics, without further phase reduction.

$$\dot{v} = v(j\Omega + c(A - |v|^2))$$

$$\begin{aligned}\dot{\Theta} &= j\Omega + c(A - |v|^2) \\ &\approx j\Omega - cA(2\Re(\Theta) - \ln(A))\end{aligned}$$

Complex Phase Analysis of Power Grid Dynamics
Jakob Niehues, Anna Büttner, Anne Riegler, FH
IEEE PowerTech 2025
https://doi.org/10.48550/arXiv.2506.22054

Power Grid as Complex Oscillators
Symmetry adapted variables

Pick physically meaningful invariants that characterize the state of the power lines:

$$v_n = e^{\Theta_n}$$

$$i_{en} = Y_e(v_n - v_m)$$

$$i_n = \sum_e i_{en}$$

$$S_n = v_n \bar{i}_n$$

Power grid model space:
ODE with the right symmetries
coupled by a complete set of invariants.

$$\dot{\Theta}_n = f^{\Theta}(\Re(\Theta_n), S_n, \overline{S}_n, x_n)$$

$$\dot{x}_n = f^{x}(\Re(\Theta_n), S_n, \overline{S}_n, x_n)$$

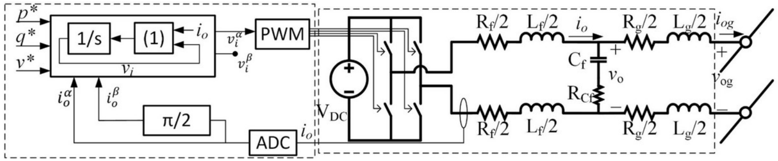
Assumption: System is somewhat close to desired state of coupling.

$$\dot{\Theta}_n = j\Omega + J^{\Theta}[\delta\Re(\Theta_n), \delta S_n, \delta\overline{S}_n, x_n]^t$$

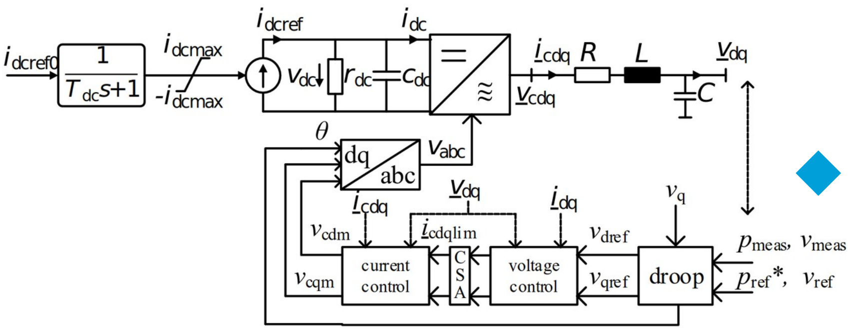
$$\dot{x}_n = J^{x}[\delta\Re(\Theta_n), \delta S_n, \delta\overline{S}_n, x_n]^t$$

Anna Büttner, FH, Complex Couplings
http://dx.doi.org/10.1103/PRXEnergy.3.013005
Raphael Kogler, Anton Plietzsch, Paul Schultz, FH, Normal Form for Grid Forming
http://dx.doi.org/10.1103/PRXEnergy.1.013008

Power Grid as Complex Oscillators
Normal Form

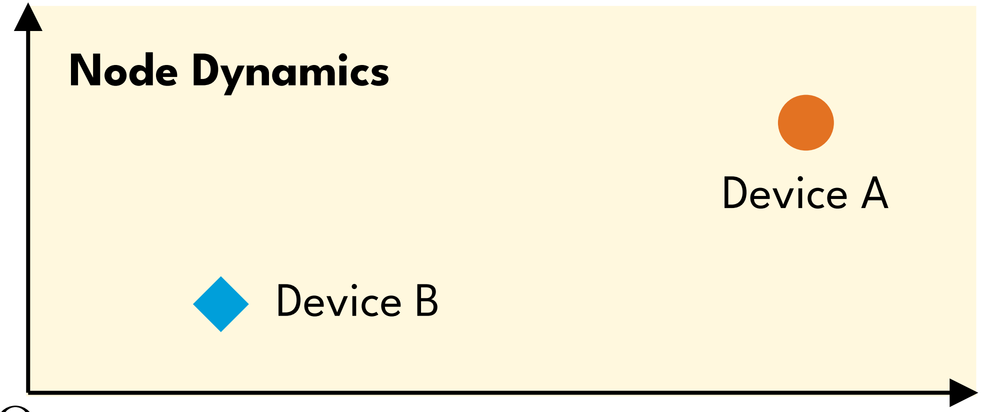


Device A



Device B

Hypothesis: The dynamical behavior of power grid actors can be captured by their linear response to variations in invariants that characterize the grids state.



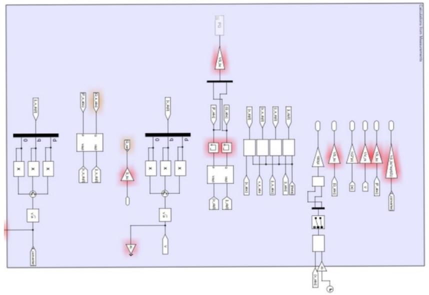
J^{Θ}, J^{x}

Power Grid as Complex Oscillators
Normal Form

Hypothesis: The dynamical behavior of power grid actors can be captured by their linear response to variations in invariants that characterize the grids state.

Lab Measurements

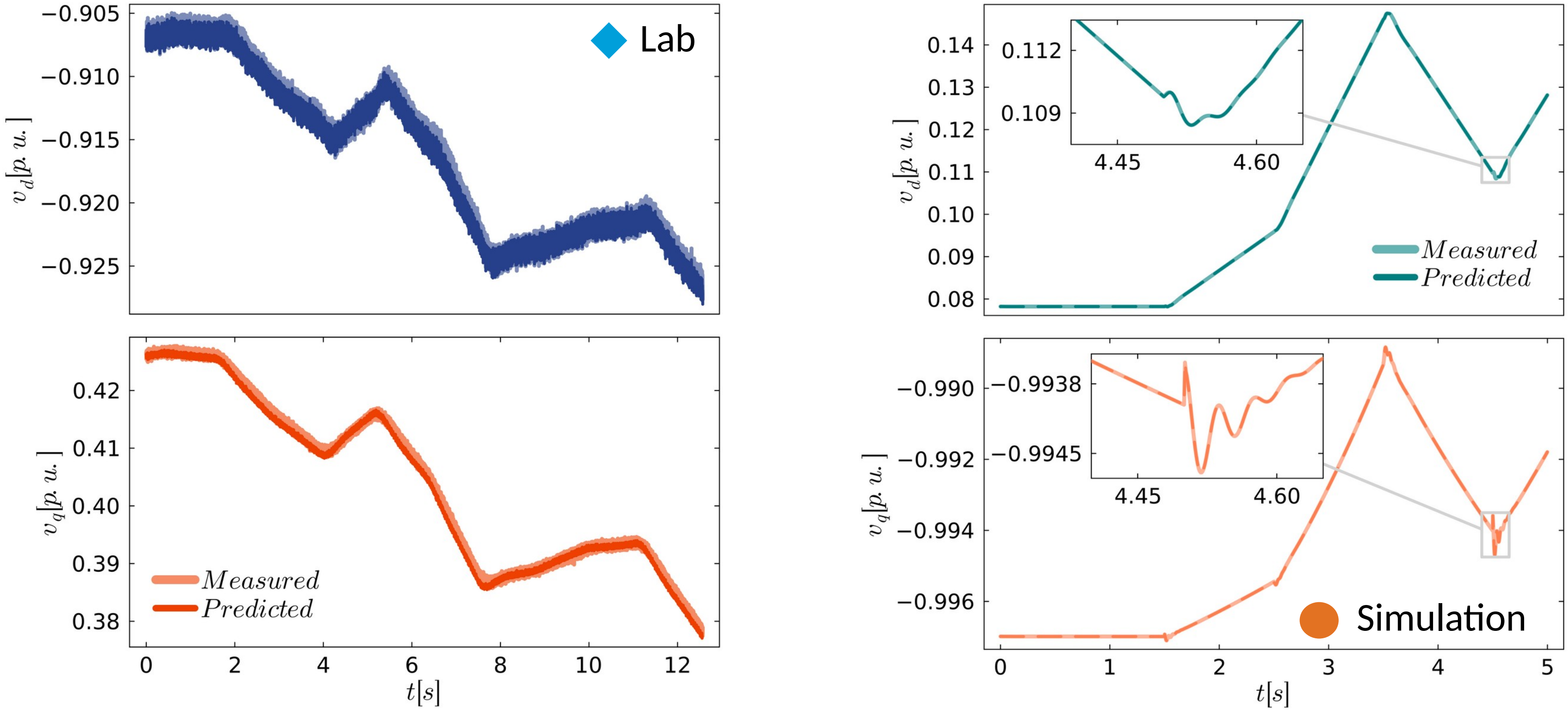
Simulations



- Lab [Sch14]
- Simulation [Col17]

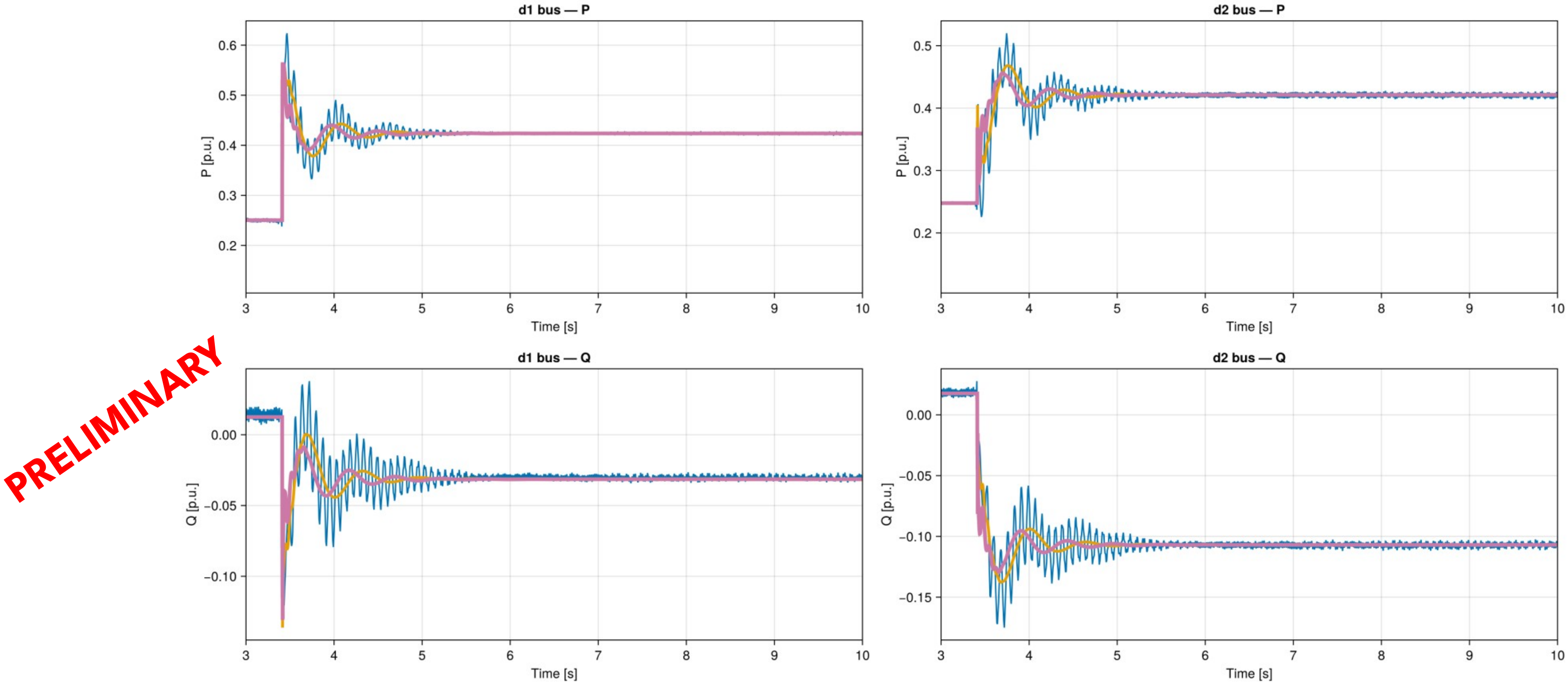
[Sch14] J.Schiffer et.al., “Conditions for stability of droop-controlled inverter-based microgrids“, Automatica, 2014
[Col17] M. Colombino, „Global phase and voltage synchronization for power inverters“, IEEE Conference on Decision and Control (CDC), 2017

Power Grid as Complex Oscillators
Normal Form



Anna Büttner; Hans Würfel; Sebastian Liemann; Johannes Schiffer; FH
"Complex-Phase, Data-Driven Identification of Grid-Forming Inverter Dynamics."
IEEE Transactions on Smart Grid (2025). https://doi.org/10.1109/TSG.2025.3591891

Power Grid as Complex Oscillators
Normal Form



Power Grid as Complex Oscillators
Another Perspective:

Power Grid Dynamics are well described by a class of generalized Stuart-Landau oscillators, parametrized by two matrices (or: an LTI system).

$$\dot{\Theta}_n = j\Omega + J^{\Theta}[\delta\Re(\Theta_n), \delta S_n, \delta\overline{S}_n, x_n]^t$$

$$\dot{x}_n = J^{x}[\delta\Re(\Theta_n), \delta S_n, \delta\overline{S}_n, x_n]^t$$

Unified mathematical form of equations also means analytic results become feasible.

Part II – Stability of networked systems

Stability of networked systems

Master Stability:

$$x \in \mathbb{R}^N \otimes \mathbb{X}$$

$$\dot{x} = (1 \otimes J_1 + L \otimes J_2)x$$

$$x^\lambda = v^\alpha \otimes z^{\lambda,\alpha}$$

$$\lambda z^{\lambda,\alpha} = (J_1 + \alpha J_2) z^{\lambda,\alpha}$$

Our setting is heterogenous in multiple ways:

Different Jacobians at different nodes.

Number of dimensions per node can vary.

Coupling not “one-dimensional”: It does not factorize.

→ We need a different way to decompose and analyze the dynamics.

Stability of networked systems

A general theory of edge weights:

$$B_e \cdot x = \begin{bmatrix} x_i \\ x_j \end{bmatrix} \text{ where } i \in e$$

$$L = \sum_e B_e^T W_e B_e$$

Standard Weighted Graph Laplacian:

$$W_e = \begin{bmatrix} K_e & -K_e \\ -K_e & K_e \end{bmatrix}$$

$$L \geq 0 \text{ if } K_e \geq 0$$

Stability of networked systems

We can think of this as a feedback connection between dynamics on the node space, and coupling in the edge space:

The stability of linear feedback connections is well studied in control theory.

$$\dot{x} = \bigoplus_n J_n^1 x - \bigoplus_n J_n^2 y$$

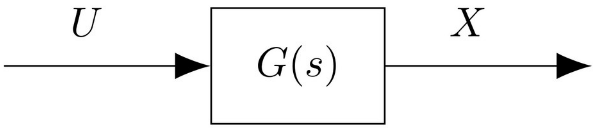
$$y = \sum_e B_e^T W_e B_e$$

Power Grid as Complex Oscillators
Stability of networked systems

$$\dot{x}(t) = -ax(t) + u(t)$$

Laplace transform with $s = \rho + i\omega$

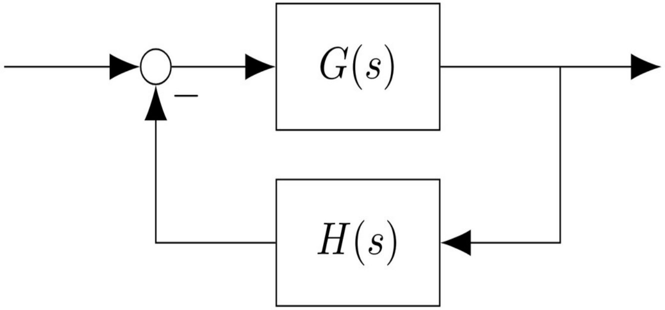
$$X(s) = \frac{1}{s+a}U(s)$$



$$G(s) = \frac{X(s)}{U(s)} = \frac{1}{s+a}$$ pole at $s = -a$

stable if $a > 0$

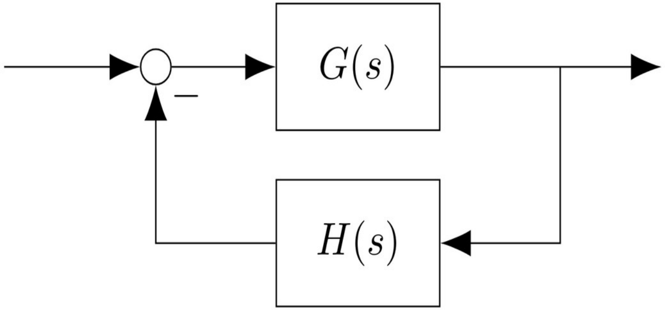
The stability of linear feedback connections is well studied in control theory.



$$G(s)(1 + H(s)G(s))^{-1}$$

Sufficient condition for stability of interconnection:

$$\{1 + G(s)H(s) | s \in j\mathbb{R}\} \cap (-\infty, 0] = \emptyset$$



We need to control the eigenvalues of this matrix valued function. E.g.:

$$|\lambda(GH)|^2 \leq \|G\|^2 \|H\|^2$$

Small Gain Theorem.

$$G(s)(1 + H(s)G(s))^{-1}$$

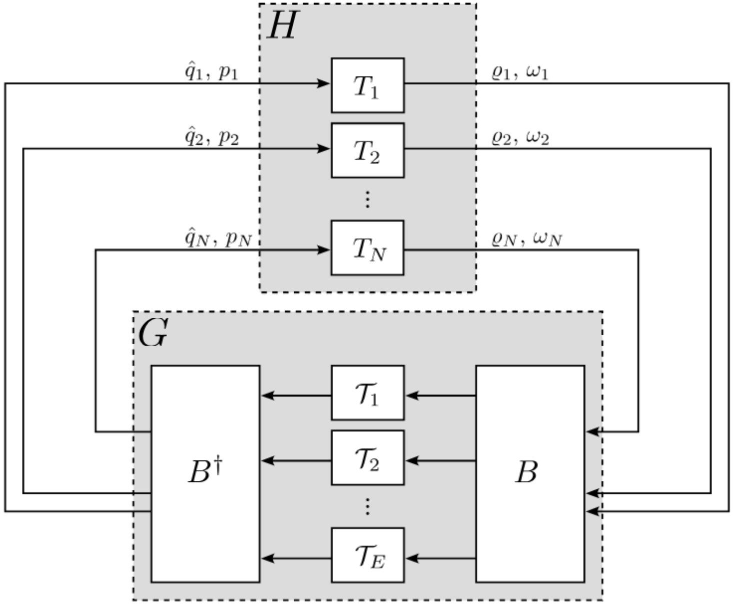
How does this look in our case?

$$\dot{x} = \bigoplus_n J_n^1 x - \bigoplus_n J_n^2 y$$

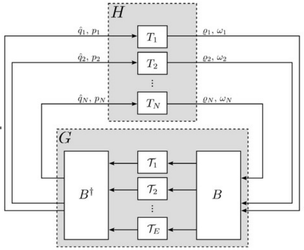
$$y = \sum_e B_e^T W_e B_e x$$

$$H(s) = \bigoplus_n (s - J_n^1)^{-1} J_n^2 = \bigoplus_n T_n(s)$$

$$G(s) = B^T \bigoplus_e \mathcal{T}_e B$$



Power Grid as Complex Oscillators
Stability of networked systems



$$H(s) = \bigoplus_n T_n(s)$$

$$G(s) = B^t \bigoplus_e \mathcal{T}_e B$$

How do we leverage this structure to control the e.v. of the product?

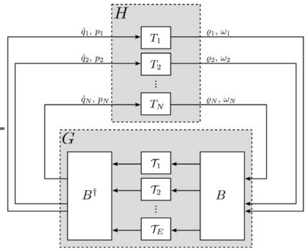
$$|\lambda(GH)|^2 \leq \|G\|^2 \|H\|^2$$

$$\|BTB^T\| \leq \|BB^T\| \|T\|$$

$$\left\| \bigoplus_n T_n \right\|^2 = \max_n \|T_n\|^2$$

Small Gain Theorem.

Power Grid as Complex Oscillators
Stability of networked systems



How do we leverage this structure to control the e.v. of the product?

$$H(s) = \bigoplus_n T_n(s)$$

$$G(s) = B^t \bigoplus_e \mathcal{T}_e B$$

$$\begin{aligned} |\lambda(GH)|^2 &\leq \|BB^T\|^2 \max_n \|T_n\|^2 \max_e \|\mathcal{T}_e\|^2 \\ &= d_{\max} \max_n \|T_n\|^2 \max_e \|\mathcal{T}_e\|^2 \end{aligned}$$

Small Gain Theorem.

Doesn't apply to diffusion with the Graph Laplacian setting!

$$\dot{x} = -Lx \implies \|T_n(s)\| = \frac{1}{|s|}$$

It's the sign, not the magnitude of edges that determines stability!

Power Grid as Complex Oscillators
Stability of networked systems

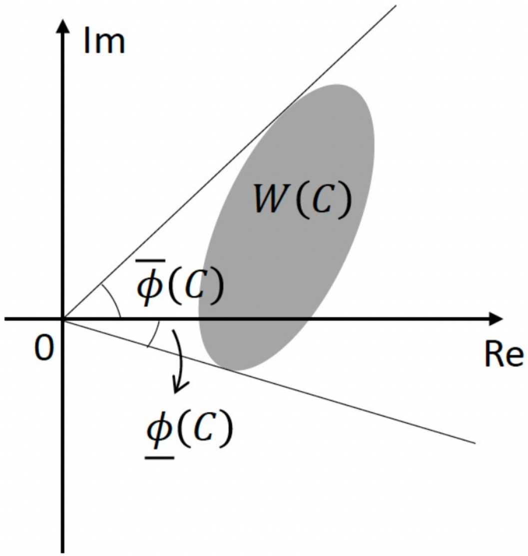
Breakthrough result:

Definition of phase of matrices that is suitable for feedback stability analysis.

On the phases of a complex matrix
Dan Wang, Wei Chen, Sei Zhen Khong, Li Qiu
Linear Algebra and its Applications (2020)
https://doi.org/10.1016/j.laa.2020.01.035

Phases:

$$W(\boldsymbol{C}) := \left\{ \boldsymbol{z}^{\dagger}\boldsymbol{C}\boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{C}^n \,,\ \boldsymbol{z}^{\dagger}\boldsymbol{z} = 1 \right\}$$



Sectorial: $\overline{\phi}(\boldsymbol{C}) - \underline{\phi}(\boldsymbol{C}) < \pi$

Power Grid as Complex Oscillators
Stability of networked systems

How do we leverage this structure to control the e.v. of the product?

$$\arg(\lambda(GH)) \leq \overline{\phi}(G) + \overline{\phi}(H)$$

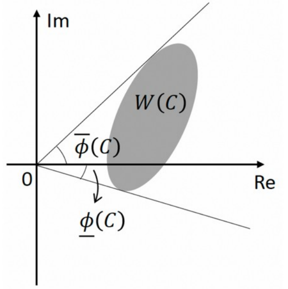
$$\overline{\phi}(BTB^T) \leq \overline{\phi}(BTB^T)$$

$$\overline{\phi}\left(\bigoplus_n T_n\right) = \max_n \overline{\phi}(T_n)$$

Small Phase Theorem!

Phases:

$$W(\boldsymbol{C}) := \left\{\boldsymbol{z}^\dagger \boldsymbol{C} \boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{C}^n, \ \boldsymbol{z}^\dagger \boldsymbol{z} = 1\right\}$$



Sectorial: $\overline{\phi}(\boldsymbol{C}) - \underline{\phi}(\boldsymbol{C}) < \pi$

$$H(s) = \bigoplus_n T_n(s)$$

$$G(s) = B^t \bigoplus_e \mathcal{T}_e B$$

Power Grid as Complex Oscillators
Stability of networked systems

How do we leverage this structure to control the e.v. of the product?

$$\arg(\lambda(GH)) \leq \overline{\phi}(G) + \overline{\phi}(H)$$

$$\overline{\phi}(BTB^T) \leq \overline{\phi}(T)$$

$$\overline{\phi}\left(\bigoplus_n T_n\right) = \max_n \overline{\phi}(T_n)$$

Small Phase Theorem!

$$\max_n \overline{\phi}(T_n) + \max_e \overline{\phi}(\mathcal{T}_e) < \pi$$

$$\dot{x} = -Lx$$

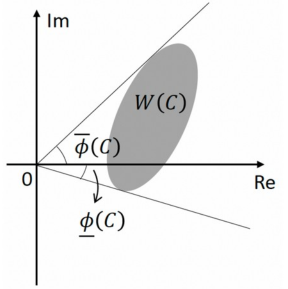
$$T_n(s) = \frac{1}{s}$$

$$\mathcal{T}_e(s) = \begin{bmatrix} K_e & -K_e \\ -K_e & K_e \end{bmatrix}$$

stable if $\Re(K_e) > 0$

Phases:

$$W(\boldsymbol{C}) := \left\{ \boldsymbol{z}^\dagger \boldsymbol{C} \boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{C}^n,\ \boldsymbol{z}^\dagger \boldsymbol{z} = 1 \right\}$$



Sectorial: $\overline{\phi}(\boldsymbol{C}) - \underline{\phi}(\boldsymbol{C}) < \pi$

$$H(s) = \bigoplus_n T_n(s)$$

$$G(s) = B^t \bigoplus_e \mathcal{T}_e B$$

Power Grid as Complex Oscillators
Stability of networked systems

Combined Gain/Phase:

stable if for every s

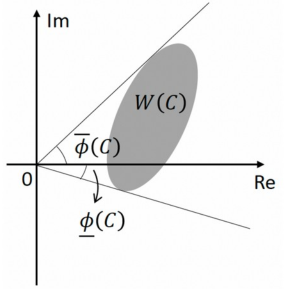
$$\text{either: } \max_n \overline{\phi}(T_n(s)) + \max_e \overline{\phi}(\mathcal{T}_e(s)) < \pi$$

$$\text{or: } d_{\max} \max_n \|T_n(s)\|^2 \max_e \|\mathcal{T}_e(s)\|^2 < 1$$

(where I have suppressed a lot of technical details, especially regarding the poles on the imaginary axis.)

Phases:

$$W(\boldsymbol{C}) := \left\{ \boldsymbol{z}^\dagger \boldsymbol{C} \boldsymbol{z} \mid \boldsymbol{z} \in \mathbb{C}^n,\ \boldsymbol{z}^\dagger \boldsymbol{z} = 1 \right\}$$



Sectorial: $\overline{\phi}(\boldsymbol{C}) - \underline{\phi}(\boldsymbol{C}) < \pi$

$$H(s) = \bigoplus_n T_n(s)$$

$$G(s) = B^t \bigoplus_e \mathcal{T}_e B$$

Kastendiek et.al. https://doi.org/10.1063/5.0249706

Stability of networked systems

How conservative are these conditions?

$$\dot{x}_n = \psi_n - \sum_{m \neq n}^{N} A_{nm} \sin(x_n - x_m)$$

$$\dot{A}_{nm} = \cos(x_n - x_m) - c_{nm} A_{nm},$$

Sufficient condition for stability:

$$|x_n^\circ - x_m^\circ| > \frac{\pi}{4}$$

This was independently known to be necessary. Phase conditions are exact here!

Phase and gain stability for adaptive dynamical networks
Nina Kastendiek, Jakob Niehues, Robin Delabays, Thilo Gross, FH
Chaos 35, 053142 (2025)
https://doi.org/10.1063/5.0249706

Stability of Power Grids

Back to our Power Grids

Proposition 1 (Small-signal stability of power grids with V-q droop). *Consider a lossless power grid with admittance matrix $\boldsymbol{Y}$ and an operating point with voltage phase angles φ_n° and magnitudes V_n°, and $\boldsymbol{T}_n(s)$ the transfer function matrices from $\hat{q}_n$, p_n, to ϱ_n and ω_n for some α_n.*
The operating point is linearly stable if $|\varphi_n^\circ - \varphi_m^\circ| < \pi/2$ for all n and m connected by a line, the $\boldsymbol{T}_n(s)$ are internally stable, and for all $s \in [0, \infty]$ it holds

$$\Re(T_n^{\varrho\hat{q}}) + \Re(T_n^{\omega p}) > 0\,, \tag{3}$$

$$\Re(T_n^{\varrho\hat{q}}) \cdot \Re(T_n^{\omega p}) > \frac{1}{4}\left|T_n^{\varrho p} + \overline{T}_n^{\omega\hat{q}}\right|^2\,, \tag{4}$$

$$\alpha_n \geq 2\sum_m \tilde{Y}_{nm}\frac{V_m^\circ}{\cos(\varphi_n^\circ - \varphi_m^\circ)}\,. \tag{5}$$

Small-Signal Stability of Power Systems With Voltage Droop
Jakob Niehues, Robin Delabays, Anna Büttner, FH
IEEE Transactions on Power Systems (2026)
https://doi.org/10.1109/TPWRS.2025.3613855

Power Grid as Complex Oscillators
Beyond degree balance

A limitation: So far we needed
undirected edges.

$$W_e = \begin{bmatrix} K_e & -K_e \\ -K_e & K_e \end{bmatrix}$$

Power Grid as Complex Oscillators
Beyond degree balance

Directed edges can be decomposed:

$$\boldsymbol{W}_e(s) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \boldsymbol{M}_e^u(s) + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \otimes \boldsymbol{M}_e^s(s),$$

A Unified Theory of Edge Weights: Stability of General Laplacian Networks from Matrix Phases and Asymmetry Rayleigh Ratios
Nina Kastendiek, Jakob Niehues, Frank Hellmann
https://arxiv.org/abs/2604.18017

Power Grid as Complex Oscillators
Beyond degree balance

$$\boldsymbol{W}_e(s) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \boldsymbol{M}_e^u(s) + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \otimes \boldsymbol{M}_e^s(s),$$

We define the ratio of superdirected and undirected coupling as the Asymmetry Rayleigh Coefficients

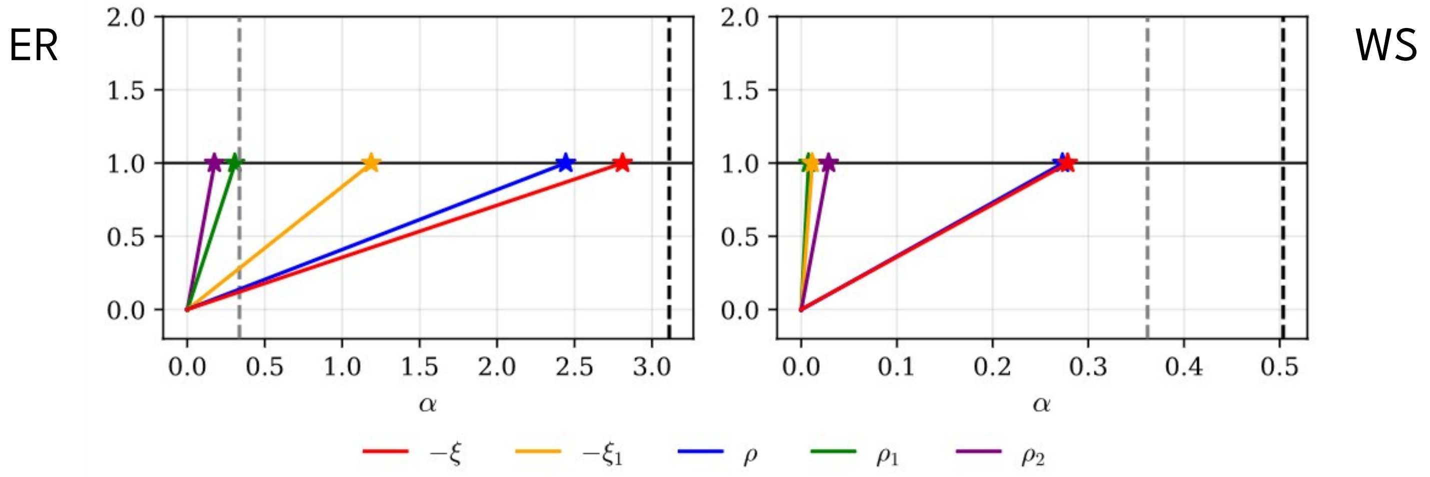
$$\rho = \max_{v \perp \ker(U)} \left| \frac{v^\dagger S v}{v^\dagger U v} \right|$$

$$\xi = \min_{v \perp \ker(U)} \Re\left(\frac{v^\dagger S v}{v^\dagger U v} \right)$$

A Unified Theory of Edge Weights: Stability of General Laplacian Networks from Matrix Phases and Asymmetry Rayleigh Ratios
Nina Kastendiek, Jakob Niehues, Frank Hellmann
https://arxiv.org/abs/2604.18017

Power Grid as Complex Oscillators
Beyond degree balance

$\dot{x} = -Lx$:



Degree balance

$$\xi_1 := \frac{\frac{1}{2}\min_n \lambda_{\min}(\boldsymbol{\Delta}_n)}{\lambda_2(\boldsymbol{L}^0)\min_e \lambda_{\min}(\boldsymbol{M}_e^u)} > -1\,,$$

Global connectivity

Coupling strength

Thank you